# Scattering of magnetic edge states

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**Abstract**

We consider a charged particle following the boundary of a two-dimensional domain because a homogeneous magnetic field is applied. We develop the basic scattering theory for the corresponding quantum mechanical edge states. The scattering phase attains a limit for large magnetic fields which we interpret in terms of classical trajectories.

## 1 Introduction

A charged particle moving in a domain $\Omega \subset \mathbb{R}^2$ under the influence of a homogeneous magnetic field $B$ may follow a skipping orbit along the boundary $\partial\Omega$. The quantum mechanical counterpart to these orbits are extended chiral states supported near $\partial\Omega$. Under certain geometric conditions these states give rise to some purely absolutely continuous spectrum [7] at energies $E$ away from the Landau levels associated with bulk states, i.e., at $E \in B \cdot \Delta$ with

$$\bar{\Delta} \cap (2\mathbb{N}+1) = \emptyset. \tag{1.1}$$

This work is about the scattering of such chiral edge states at a bent of an otherwise straight boundary $\partial\Omega$. While they, being chiral, never backscatter, they acquire an additional phase as compared to a particle following a straight boundary of the same length. The main result is, that this phase is proportional to the bending angle but independent of the (large) $B$ field. We remark that the scattering of edge states is at the basis of some theories of the quantum Hall effect [4].

The precise formulation of the setup and of the results requires some preliminaries. We consider a simply connected domain $\Omega \subset \mathbb{R}^2$ with oriented boundary $\partial\Omega$ consisting of a single, unbounded smooth curve $\gamma \in C^4(\mathbb{R})$ parameterized by arc length $s \in \mathbb{R}$. We assume that $\gamma$ is eventually straight in

the sense that the curvature $\kappa(s) = \dot\gamma(s)\wedge\ddot\gamma(s) \in \mathbb{R}$, $(\cdot = \mathrm{d}/\mathrm{d}s)$, is compactly supported. The bending angle

$$\theta := \int_{-\infty}^{\infty} \kappa(s)\,\mathrm{d}s$$

takes values in $[-\pi, \pi]$ and we assume

$$\theta \neq \pi, \tag{1.2}$$

which ensures that $\Omega$ contains a wedge of positive opening angle.

Since the cyclotron radius, and hence the lateral extent of an edge state, scales as $B^{-1/2}$, it will be notationally convenient to represent the homogeneous field as $B = \beta^2$. The Hamiltonian is

$$H = B^{-1}(-i\nabla - BA(x))^2 = (-i\beta^{-1}\nabla - \beta A(x))^2 \tag{1.3}$$

on $L^2(\Omega)$ with Dirichlet boundary conditions on $\partial\Omega$. Here $A : \Omega \to \mathbb{R}^2$ is a gauge field producing a unit magnetic field, $\partial_1 A_2 - \partial_2 A_1 = 1$. This is the usual magnetic Hamiltonian except for a rescaling of energy, which is now measured in units of Landau levels spacings. This, or the equivalent rescaling of time, does not affect the scattering operator, but will simplify its analysis.

As the dynamics of the edge states is effectively one-dimensional, it is natural to eliminate the gauge field from its description. For the 2-dimensional system this means that we restrict to gauges with $A_{\|} = 0$ on $\partial\Omega$, i.e.,

$$A(\gamma(s)) \cdot \dot\gamma(s) = 0. \tag{1.4}$$

A particle moving in a half-plane $\Omega_0 = \mathbb{R} \times \mathbb{R}_+ \ni (s, u)$ will serve as a model for the asymptotic dynamics, both in the past (or at $s \to -\infty$) and in the future (or at $s \to +\infty$). We denote the corresponding Hamiltonian on $\mathcal{H}_0 := L^2(\Omega_0)$ by

$$H_0 := (-i\beta^{-1}\partial_s + \beta u)^2 + (-i\beta^{-1}\partial_u)^2, \tag{1.5}$$

where we have used the Landau gauge $A = (-u, 0)$.

To serve as scattering asymptotes, states in $L^2(\Omega_0)$ have to be identified with states in $L^2(\Omega)$. To this end we introduce the tubular map:

$$\begin{aligned} \mathcal{T} : \Omega_0 &\to \mathbb{R}^2 \\ x(s,u) \equiv \mathcal{T}(s,u) &= \gamma(s) + u\varepsilon\dot\gamma(s), \end{aligned} \tag{1.6}$$

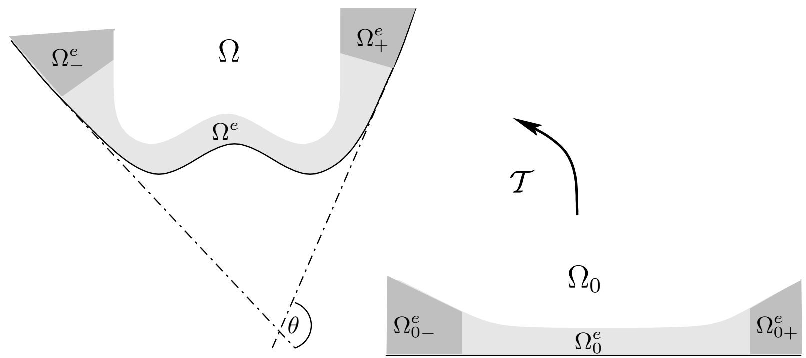



Figure 1: Left: The domains $\Omega$, $\Omega^e$, $\Omega^e_\pm$.

Figure 2: Right: The domains $\Omega_0$, $\Omega^e_0$, $\Omega^e_{0\pm}$.

where $\varepsilon = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is the rotation by $\pi/2$, and hence $\varepsilon\dot\gamma(s)$ the inward normal. The map $\mathcal{T}$ is injective on

$$\Omega^e_0 := \{(s,u) \in \Omega_0 \mid s \in \mathbb{R},\, 0 \le u < w(s)\},$$

with Jacobian $|\det \mathrm{D}\mathcal{T}| = 1 - u\kappa(s)$ uniformly bounded away from zero, for some sufficiently small positive, continuous width function $w(s)$. Due to condition (1.2) we may take it so that

$$w(s) \ge c_1 + c_2|s| \tag{1.7}$$

for some $c_1$, $c_2 > 0$. The map (1.6) provides coordinates $(s,u)$ on the image $\Omega^e := \mathcal{T}(\Omega^e_0) \subset \Omega$ (Fig. 1, 2). Not all of $\Omega^e_0$ is essential for the sought identification, but only its tails near $s = \pm\infty$,

$$\Omega^e_{0\pm} := \{(s,u) \in \Omega^e_0 \mid \pm s > C\}.$$

For large enough $C$ the tubular map is Euclidean if restricted to $\Omega^e_{0\pm}$, since $\operatorname{supp}\ddot\gamma$ is compact. To make the dynamics of (1.3) and (1.5) comparable, we assume that

$$A(x) = (-u, 0), \qquad (x \in \Omega^e_\pm) \tag{1.8}$$

w.r.t. the Euclidean coordinates $(s,u)$ in $\Omega^e_\pm := \mathcal{T}(\Omega^e_{0\pm})$. This does not fix the potential $A$ outside of $\Omega^e_- \cup \Omega^e_+$ beyond the condition (1.4). Any residual gauge transformation $A \to A + \nabla\chi$ in $\Omega$ consistent with these requirements

has $\chi(x)$ constant in $\Omega_-^e \cup \Omega_+^e$. In fact, $\chi(x)$ takes constant values $\chi_\pm$ separately on $\Omega_\pm^e$, and

$$\chi_+ - \chi_- = \int\limits_{-\infty}^{\infty} \nabla\chi(\gamma(s)) \cdot \dot\gamma(s)\, \mathrm{d}s = 0. \tag{1.9}$$

The asymptotic Hilbert space $L^2(\Omega_0)$ is now mapped into $L^2(\Omega)$ by means of

$$J : L^2(\Omega_0) \to L^2(\Omega)$$
$$(J\psi)(x) = \begin{cases} j(u - w(s))\psi(s,u), & \text{if } x = x(s,u) \in \Omega^e, \\ \quad 0, & \text{otherwise}, \end{cases} \tag{1.10}$$

where $j \in C^\infty(\mathbb{R})$, $j \leq 1$ is such that

$$j(u) = \begin{cases} 1, & u \leq -2w_0, \\ 0, & u \geq -w_0, \end{cases} \tag{1.11}$$

for some $w_0$. The purpose of the transition function $j$ is to make $J\psi$ as smooth as $\psi$. If $w_0$ is large enough, $\operatorname{supp} J\psi \subset \Omega_-^e \cup \Omega_+^e$; if, on the other hand, $w_0$ is small enough, we have $J\psi(x) = \psi(s,0)$ for all $x = x(s,0) \in \partial\Omega$.

The first result establishes the usual properties of scattering.

**Theorem 1.** *The wave operators*

$$W_\pm : L^2(\Omega_0) \to L^2(\Omega)$$
$$W_\pm := \operatorname*{s-lim}_{t\to\pm\infty} e^{iHt} J e^{-iH_0 t}$$

*exist and are complete:*

$$\operatorname{Ran} W_\pm = P_{\mathrm{ac}}(H)\mathcal{H}.$$

*Moreover, $W_\pm$ are isometries and do not depend on the choice of $w$, $j$ in the definition of $J$.*

*Remark* 1. Under a residual gauge transformation the wave operators transform as

$$W_\pm \to e^{i(\chi_\pm - \chi(x))} W_\pm,$$

implying by (1.9) that the scattering operator $W_+^* W_-$ is invariant.

We next consider the limit where $\beta$ grows large while the energy, rescaled as in (1.3), is kept fixed. The limit of the scattering operator is thus best formulated in a scheme where edge states with fixed energy are displayed as being independent of $\beta$. The domain $\Omega_0$ is invariant under scaling

$$x \to \beta x \tag{1.12}$$

and the Hamiltonian transforms as

$$H_0 \cong -\partial_u^2 + (-i\partial_s + u)^2, \tag{1.13}$$

which shows that the spectrum of $H_0$ is independent of $\beta$. Let $\mathcal{H}_T := L^2(\mathbb{R}_+, \mathrm{d}u)$ be the space of transverse wave functions, on which $-\partial_u^2$ acts with a Dirichlet boundary conditions at $u = 0$. The translation invariance in $s$ of (1.13) calls for the (inverse) Fourier transform

$$\mathcal{F}_\beta : \int^\oplus \mathcal{H}_T \,\mathrm{d}k \cong L^2(\mathbb{R}, \mathcal{H}_T) \;\to\; L^2(\Omega_0), \qquad \psi = \int^\oplus \psi(k)\,\mathrm{d}k \;\mapsto\; \mathcal{F}_\beta \psi,$$

$$(\mathcal{F}_\beta \psi)(k) = \operatorname*{l.i.m.}_{K\to\infty} \frac{\beta^{1/2}}{(2\pi)^{1/2}} \int_{-K}^{K} e^{i\beta k s} \mathcal{D}_\beta \psi(k)\,\mathrm{d}k, \tag{1.14}$$

where the scaling of $x = (s, u)$ has been incorporated for $u$ by means of

$$\mathcal{D}_\beta : \mathcal{H}_T \;\to\; \mathcal{H}_T, \qquad (\mathcal{D}_\beta \psi)(u) = \beta^{1/2} \psi(\beta u), \tag{1.15}$$

and for $s$ explicitly in the integral. (It is, in a precise sense, a Bochner integral of $\mathcal{H}_T$-valued functions [1, Sec. 1.1,Sec. 1.8]). Then

$$\mathcal{F}_\beta^{-1} H_0 \mathcal{F}_\beta = \widehat{H}_0 := \int^\oplus H_0(k)\,\mathrm{d}k, \qquad H_0(k) = -\partial_u^2 + (k+u)^2. \tag{1.16}$$

The fiber $H_0(k)$, see [5], has simple, discrete spectrum $\{E_n(k)\}_{n\in\mathbb{N}}$ with projections denoted as $P_n(k)$. The energy curve $E_n(k)$, called the $n$-th deformed Landau level, is a smooth function of $k$ increasing from $2n+1$ to $+\infty$ for $k \in (-\infty, \infty)$ with $E_n'(k) > 0$. The corresponding normalized eigenvectors by $\psi_n(k)$ may be taken as smooth functions (in $\mathcal{H}_T$-norm) of $k$, though the choice is affected by the arbitrariness of their phase,

$$\psi_n(k) \mapsto \; e^{i\lambda_n(k)} \psi_n(k). \tag{1.17}$$

They decay exponentially in $u$ (see Lemma 7).

In this scheme the scattering operator is

$$S = \mathcal{F}_\beta^{-1} W_+^* W_- \mathcal{F}_\beta \, : \int^\oplus \mathcal{H}_T \,\mathrm{d}k \to \int^\oplus \mathcal{H}_T \,\mathrm{d}k, \tag{1.18}$$

and becomes independent of the magnetic field if large:

**Theorem 2.** *We have*

$$\operatorname*{s-lim}_{\beta\to\infty} S = S_\phi \tag{1.19}$$

*with*

$$\begin{aligned} S_\phi &= \int^\oplus \sum_n e^{i\phi_n(k)} P_n(k)\,\mathrm{d}k, \\ \phi_n(k) &= -\frac{E_n^{(1)}(k)}{E_n'(k)}\theta, \\ E_n^{(1)}(k) &= \langle \psi_n(k)\,,\, H_1(k)\psi_n(k)\rangle, \\ H_1(k) &= u^3 + 3u^2 k + 2uk^2. \end{aligned} \tag{1.20}$$

*More precisely, if energies are restricted to any open interval $\Delta$ between Landau levels, as in (1.1), the limit (1.19) holds in norm: for any $\varepsilon > 0$ there is $C_{\Delta,\varepsilon}$ such that*

$$\left\| (S - S_\phi) E_\Delta(\widehat{H}_0) \right\| \le C_{\Delta,\varepsilon} \beta^{-1+\varepsilon}. \tag{1.21}$$

*Remark* 2. $E_n^{(1)}(k)$ is the first order correction to the eigenvalue $E_n(k)$ under the (singular) perturbation $\beta^{-1}\kappa(s)H_1(k)$ of $H_0(k)$ due to the curvature of the boundary.

We conclude with some comments about the origin of the phase $\phi_n$. The Hamiltonian (1.3) results from the quantization of mixed systems [10] in the sense that it may be regarded as the quantization over the phase space $\mathbb{R}^2 \ni (s,k)$ of the classical symbol

$$H(s,k) = H_0(k) + \beta^{-1}\kappa(s)H_1(k), \tag{1.22}$$

which formally takes values in the operators on $\mathcal{H}_T$. Typical WKB solutions for such systems have a phase consisting of a dynamical part of $\mathrm{O}(\beta)$ followed by a geometric part, namely the Berry and Rammal-Wilkinson phases, $\gamma_{\mathrm{B}} + \gamma_{\mathrm{RW}}$, which are of $\mathrm{O}(1)$. The scattering operator $S$ discounts from this the phase that pertains to the principal symbol $H_0(k)$ alone. The phase left

over thus stems from the sub-principal symbol only, with the two parts now suppressed by a factor $\beta^{-1}$. The phase (1.20) is thus dynamical — despite its connection with the geometry of $\Omega$ —, while for the geometric ones we find $\beta^{-1}$ times

$$\gamma_{\mathrm{B}}(s,k) = \kappa(s)\frac{E_n^{(1)}(k)}{E_n'(k)}\,\mathrm{Im}\,\langle\psi_n(k)\,,\,\partial_k\psi_n(k)\rangle\,, \tag{1.23}$$

$$\gamma_{\mathrm{RW}}(s,k) = -\gamma_{\mathrm{B}}(s,k) + \frac{\kappa(s)}{E_n'(k)}\,\mathrm{Im}\,\langle H_1(k)\psi_n(k)\,,\,\partial_k\psi_n(k)\rangle\,. \tag{1.24}$$

In the next section we give a heuristic interpretation of the edge states and of the scattering phase $\phi_n(k)$ in terms of classical orbits bouncing at the boundary. Related considerations are found in [8]. Readers more interested in the proofs may proceed without loss to Sects. 3, 4. Higher order corrections like (1.23, 1.24) are discussed in Sect. 5.

## 2 Classical trajectories and scattering phase

The Hamiltonian

$$H_0 = (\beta^{-1}p_s + \beta u)^2 + \beta^{-2}p_u^2,$$

which is the classical counterpart to (1.5), has circular trajectories for which radius $r > 0$ and velocity $v \in \mathbb{R}^2$ are in the fixed relation $r = |v|/2$. Some of them bounce along the edge of the half-plane. Their shape may be parameterized in various ways: (i) By the ratio

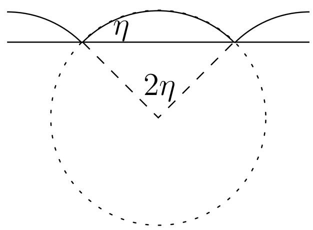
$$\frac{k}{r} = \cos\eta \tag{2.1}$$

between the distance $k = \beta^{-2}p_s$ of the guiding center to $\partial\Omega_0$ (negative, if inside $\Omega_0$) and the radius $r$. This is also expressed through the angle $\eta$ between the boundary and the arc, see Fig. 3. (ii) By the ratio

$$\frac{v_\parallel}{|v|} = \frac{\sin\eta}{\eta} \tag{2.2}$$

between the average velocity $v_\parallel$ along the edge and the (constant) velocity $|v|$ or, equivalently, between the length $2r\sin\eta$ of the chord and $2r\eta$ of the arc in Fig. 3.

We now turn to the quantum state $e^{iks}\psi_n(k)$ for $\beta = 1$, cf. (1.14). On the basis of (2.1) it may be associated, at least asymptotically for large $n$, with



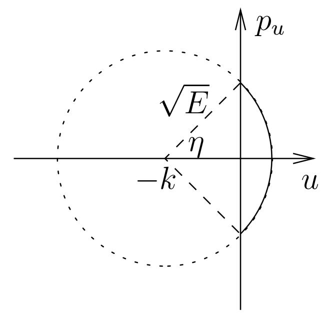




Figure 3: Left: a bouncing trajectory

Figure 4: Right: the phase space of transversal motion

a classical trajectory of shape $\eta$ if

$$k_n = \sqrt{E_n(k_n)}\cos\eta. \tag{2.3}$$

The same conclusion is reached on the basis of (2.2) if $v_\parallel$ is identified with the group velocity $E_n'(k)$, as we presently explain. The phase space $\mathbb{R}_+ \times \mathbb{R} \ni (u, p_u)$ underlying $H_0(k)$ is shown in Fig. 4, together with a trajectory of energy $(k+u)^2 + p_u^2 = E$. Let $A(E, k)$ be the area of the cap inside this trajectory. The Bohr-Sommerfeld condition, whose asymptotic validity we take for granted, states that $A(E_n(k), k) = 2\pi n$, $(n \in \mathbb{N})$, and derivation w.r.t. $k$ yields

$$\frac{\partial A}{\partial E} E_n'(k) + \frac{\partial A}{\partial k} = 0.$$

Using that $-\partial A/\partial k$ is the length of the chord in Fig. 4, and $\partial A/\partial\sqrt{E} = 2\sqrt{E}(\partial A/\partial E)$ that of the arc we find

$$\frac{v_\parallel}{|v|} = \frac{E_n'(k)}{2\sqrt{E_n(k)}} = \frac{\sin\eta}{\eta}, \tag{2.4}$$

provided $k = k_n$ is chosen as in (2.3). The energy is then $E_n(k_n) \propto n$ and the radius before the scaling (1.12) is given by $r_n^2 = \beta^{-2}E_n(k_n)$.

In light of this correspondence we shall discuss the motion along a curved boundary. The semiclassical limit, $n \gg 1$, and the limit of small curvature, $\kappa(s)r_n \ll 1$, are consistent as long as $1 \ll n \ll \beta^2\kappa^{-2}$, i.e., for large magnetic fields. We again first deal with the classical particle, whose incidence angle

$\eta$ may now slightly change from hit to hit. Let

$$G(s,s';\kappa)=\int_\gamma p\cdot \mathrm{d}x$$

be the (reduced) action along one of the two arcs $\gamma$ of radius $r$ joining neighboring collision points $s$ and $s'$ along the boundary of curvature $\kappa(\cdot)$ (provided they are close enough, so that the arcs exist). With $p=\beta^2(v/2+A)$ we obtain [2]

$$G(s,s';\kappa)=\beta^2(r\mathcal{L}-\mathcal{A}),$$

where $\mathcal{L}$ is the length of the arc $\gamma$ and $\mathcal{A}$ the area between the arc and the boundary $\partial\Omega$. In fact,

$$\frac{1}{2}\int_\gamma v\cdot \mathrm{d}x=\frac{|v|}{2}\int_\gamma \frac{\mathrm{d}x}{\mathrm{d}\sigma}\cdot \mathrm{d}x=r\int_\gamma \mathrm{d}\sigma,$$

where $\sigma$ is the arc length along $\gamma$; and, by Stokes' theorem, $\int_\gamma A\cdot \mathrm{d}x = -\mathcal{A}$, because the arc is traversed clockwise and because of (1.4). We next consider an arc starting at $s$ with angle $\eta$ and look for the dependence of $s'-s$, $\eta'-\eta$ and $G(s,s';\kappa)$ up to first order in a small curvature $\kappa$. Elementary considerations show that

$$\begin{aligned}
&\delta(s'-s)\approx -\kappa r^2\sin 2\eta, \qquad \delta(\eta'-\eta)\approx 0,\\
&\delta\mathcal{L}\approx -2\kappa r^2\sin\eta, \qquad \delta\mathcal{A}\approx -\frac{4}{3}\kappa r^3\sin^3\eta,\\
&\delta G(s,s';\kappa)\approx -2\beta^2 r^3\kappa\sin\eta\big(1-\frac{2}{3}\sin^2\eta\big),
\end{aligned}$$

where $\kappa=\kappa(\tilde{s})$ for any $\tilde{s}$ between $s$ and $s'$. We then take a number $m$ of hops $s_i=s_i[\kappa]$, $(i=0,\dots m)$ sufficient to cover the bent $\operatorname{supp}\kappa$. Using $s_i-s_{i-1}=2r\sin\eta$ for $\kappa\equiv 0$, we compute in the small curvature limit

$$\delta(s_m-s_0)=\sum_{i=1}^{m}\frac{\delta(s_i-s_{i-1})}{s_i-s_{i-1}}(s_i-s_{i-1})\approx -r\frac{\sin 2\eta}{2\sin\eta}\int_{-\infty}^{\infty}\kappa(s)\,\mathrm{d}s=-r\theta\cos\eta,$$

$$\delta\sum_{i=1}^{m}G(s_{i-1},s_i;\kappa)\approx -\beta^2r^2\theta\big(1-\frac{2}{3}\sin^2\eta\big).$$

An incoming quantum wave $e^{i\beta ks}\mathcal{D}_\beta\psi(k)$ should therefore gather an additional phase

$$\phi_n(k) = -\beta k\delta(s_m - s_0) + \delta\sum_{i=1}^{m} G(s_{i-1}, s_i; \kappa)$$

as compared to one following a straight boundary of the same length. With (2.3) we find

$$\phi_n(k_n) = \beta r_n\theta\sqrt{E_n(k_n)}\cos^2\eta - \beta^2 r_n^2\theta\bigl(1 - \frac{2}{3}\sin^2\eta\bigr) = -\frac{1}{3}\theta E_n(k_n)\sin^2\eta. \tag{2.5}$$

On the other hand, the phase $\phi_n(k_n)$ may be computed from (1.20). Since the trajectory in Fig. 4 is traversed at a uniform rate, expectations w.r.t. $\psi_n(k_n)$ reduce in the limit to integrations w.r.t. $(2\eta)^{-1}\mathrm{d}\alpha$, where a point on the arc is represented by its angle $\alpha \in [-\eta, \eta]$ as seen from the center of the circle. We rewrite $\sqrt{E}\cos\alpha = k+u =: u'$ and $u^3+3u^2k+2uk^2 = u'(u'^2-k^2)$, use

$$\frac{1}{2\eta}\int_{-\eta}^{\eta}\cos\alpha\,\mathrm{d}\alpha = \frac{\sin\eta}{\eta}, \qquad \frac{1}{2\eta}\int_{-\eta}^{\eta}\cos^3\alpha\,\mathrm{d}\alpha = \frac{\sin\eta}{\eta}\bigl(1 - \frac{1}{3}\sin^2\eta\bigr),$$

and obtain

$$E_n^{(1)}(k) \approx E_n(k_n)^{3/2}\frac{\sin\eta}{\eta}\bigl(1 - \frac{1}{3}\sin^2\eta - \cos^2\eta\bigr) = \frac{2}{3}E_n(k_n)^{3/2}\frac{\sin^3\eta}{\eta},$$
$$\phi_n(k_n) \approx -\frac{1}{3}\theta E_n(k_n)\sin^2\eta,$$

where we used (2.4) in the last step. The result is in agreement with (2.5).

## 3 Existence and completeness of wave operators

Existence and completeness of the wave operators $W_\pm$ follow in a rather standard way from propagation estimates for the dynamics $e^{-iHt}$ and $e^{-iH_0t}$.

Such an estimate is established in the second part of the following lemma. It depends on a Mourre estimate [7], which in turn rests on a geometric property discussed in the first part:

**Lemma 1.** *1. There is a function $\sigma \in C^2(\bar\Omega)$ extending arc length from $\partial\Omega$ to $\Omega$, i.e., $\sigma(\gamma(s)) = s$ for $s \in \mathbb{R}$, satisfying*

$$\|\partial_i\sigma\|_\infty < \infty, \quad \|\partial_i\partial_j\sigma\|_\infty < \infty. \tag{3.1}$$

*2. For any $\varepsilon > 0$, $\alpha > 1/2$ and $\Delta$ as in (1.1):*

$$\int_{-\infty}^{\infty} \left\| \langle\sigma\rangle^{-\alpha} e^{-iHt} E_\Delta(H)\psi \right\|^2 \mathrm{d}t \le C_{\Delta,\alpha} \beta^{1+\varepsilon} \|\psi\|^2 \tag{3.2}$$

*with $C_{\Delta,\alpha}$ independent of large enough $\beta$.*

*Proof.* 1. On $\Omega_0$ we may choose the following extension of arc length:

$$\sigma_0(s,u) := \frac{s}{w(s)}(w(s)-u)j(u-w(s)). \tag{3.3}$$

It satisfies (3.1) and is supported on $\Omega_0^e$. We therefore obtain an extension of arc length $\sigma(x)$ from $\partial\Omega$ to $\Omega$ by transforming $\sigma_0$ under the tubular map:

$$\sigma(x) := \begin{cases} \sigma_0(s,u) & \text{if } x = x(s,u) \in \Omega^e, \\ 0 & \text{otherwise.} \end{cases}$$

$\sigma$ satisfies (3.1) because $\sigma_0$ is an extension of arc length on $\Omega_0$, $\sigma$ is supported on $\Omega^e$ and the inverse tubular map has bounded first and second derivatives on $\Omega^e$. The extension of $\sigma$ by zero to the complement of $\Omega^e$ is smooth by construction of $j$.

2. To better display the dependence on $\beta$ of some of the bounds below we scale $\Omega$ to $\tilde{\Omega} = \beta\Omega$, so that $H \cong \tilde{H}$, where

$$\tilde{H} = (-i\nabla - \tilde{A})^2,$$

on $L^2(\tilde{\Omega})$ with $\tilde{A}(x) = \beta A(x/\beta)$ corresponding to a unit magnetic field. The corresponding extension of arc length from part (1) is $\tilde{\sigma}(x) = \beta\sigma(x/\beta)$. We claim that for given $E \notin 2\mathbb{N}+1$

$$\left\| [\tilde{H}\,,\,\tilde{\sigma}](\tilde{H}+i)^{-1} \right\| \le C, \tag{3.4}$$

$$\left\| [[\tilde{H}\,,\,\tilde{\sigma}]\,,\,\tilde{\sigma}] \right\| \le C, \tag{3.5}$$

$$E_{\tilde{\Delta}}(\tilde{H}) i[\tilde{H}\,,\,\tilde{\sigma}] E_{\tilde{\Delta}}(\tilde{H}) \ge c E_{\tilde{\Delta}}(\tilde{H}) \tag{3.6}$$

with $C, c > 0$ and an open interval $\tilde{\Delta} \ni E$, all independent of $\beta$ large. Indeed, (3.4, 3.5) follow from

$$i[\tilde{H}\,,\,\tilde{\sigma}] = (-i\nabla - \tilde{A})\cdot\nabla\tilde{\sigma} + \nabla\tilde{\sigma}\cdot(-i\nabla - \tilde{A}),$$
$$i[i[\tilde{H}\,,\,\tilde{\sigma}]\,,\,\tilde{\sigma}] = 2(\nabla\tilde{\sigma})^2,$$

and (3.6) has been shown in connection with the proof of Thm. 3 in [7]. The bounds (3.4-3.6) now imply [9] for $\alpha > 1/2$:

$$\int_{-\infty}^{\infty} \left\| \langle\tilde{\sigma}\rangle^{-\alpha} e^{-i\tilde{H}t} E_{\tilde{\Delta}}(\tilde{H})\psi \right\|^2 \mathrm{d}t \le C\|\psi\|^2 .$$

Undoing the unitary scale transformation, this amounts to:

$$\beta^{-2\alpha} \int_{-\infty}^{\infty} \left\| (\sigma^2 + \beta^{-2})^{-\alpha/2} e^{-iHt} E_{\tilde{\Delta}}(H)\psi \right\|^2 \mathrm{d}t \le C\|\psi\|^2 .$$

Using a covering argument for $\Delta$, this proves

$$\int_{-\infty}^{\infty} \left\| \langle\sigma\rangle^{-\alpha} e^{-iHt} E_{\Delta}(H)\psi \right\|^2 \mathrm{d}t \le C_\alpha \beta^{2\alpha} \|\psi\|^2 , \tag{3.7}$$

for $\beta \ge 1$, which may be assumed without loss. For $\alpha \le (1+\varepsilon)/2$ the claim follows from $\beta^{2\alpha} \le \beta^{1+\varepsilon}$. It then extends to $\alpha > (1+\varepsilon)/2$ because the l.h.s of (3.7) is decreasing in $\alpha$. □

*Remark* 3. The bound (3.2) may be understood in simple terms. The velocity of a particle tangential to the boundary is $i[H, s] = \beta^{-1}(-i\beta^{-1}\nabla - \beta A(x)) \cdot \nabla s = O(\beta^{-1})$, assuming its energy $H$ lies in $\Delta$. It therefore takes the particle a time $O(\beta)$ to traverse a fixed piece of the boundary such as the bent. Eq. (3.2) is stating just this, up to a multiplicative error $O(\beta^{\varepsilon})$.

We shall prove existence and completeness of the wave operators $W_\pm$ by local Kato smoothness. More precisely by [13, Thm. XIII.31] or, with more detail, by [14, Sect. 4.5, Thm. 1, Cor. 2, Rem. 3, Thm. 6] all of Thm. 1, except for the uniqueness statement, is implied by the following lemma:

**Lemma 2.** *1. $J$ maps $\mathcal{D}(H_0)$ into $\mathcal{D}(H)$. Moreover*

$$HJ - JH_0 = \sum_{i=1}^{2} A_i^* M_i A_i^0 , \tag{3.8}$$

*where $A_i^{(0)}$ are $H_{(0)}$-bounded and $H_{(0)}$-smooth on $\Delta$, and $M_i$ are bounded operators, ($i = 1, 2$, $(0) = 0$ or its omission).*

*2.*

$$\operatorname*{s-lim}_{t\to\pm\infty} (1 - JJ^*) e^{-iHt} E_\Delta(H) = 0. \tag{3.9}$$

*Proof.* 1. For $C$ large enough, $|\sigma_0(s,u)| > C$ implies $j(u-w(s)) = 1$. In fact, if $j(u-w(s)) < 1$ we have $u-w(s) > -2w_0$ and therefore, see eq. (3.3),

$$|\sigma_0(s,u)| = \frac{|s|}{w(s)}(w(s)-u)j(u-w(s)) \tag{3.10}$$

is bounded by $2w_0 \sup_s |s|/w(s)$, which is finite by (1.7). By (3.10) we also see that $|\sigma_0(s,u)| > C$ implies that $|s|$ is large. These two implications, together with (1.8), show that $(HJ-JH_0)F(|\sigma_0| > C) = 0$, where $F(x \in A)$ is the characteristic function of the set $A$. Together with a similar relation for $\sigma$ instead of $\sigma_0$ we obtain

$$HJ - JH_0 = \chi(HJ - JH_0)\chi_0, \tag{3.11}$$

where $\chi_{(0)} = F(|\sigma_{(0)}| \le C)$.

Eq. (3.11) may be written in the form (3.8) with

$$\begin{aligned}
A_1 &= \langle\sigma\rangle^{-\alpha}(H-i),\\
M_1 &= \langle\sigma\rangle^{\alpha}(H+i)^{-1}\chi HJ\chi_0\langle\sigma_0\rangle^{\alpha},\\
A_1^0 &= \langle\sigma_0\rangle^{-\alpha},\\
A_2 &= \langle\sigma\rangle^{-\alpha},\\
M_2 &= -\langle\sigma\rangle^{\alpha}\chi JH_0\chi_0(H_0+i)^{-1}\langle\sigma_0\rangle^{\alpha},\\
A_2^0 &= \langle\sigma_0\rangle^{-\alpha}(H_0+i).
\end{aligned}$$

The claimed properties about the $A_i^{(0)}$ hold true by (3.2) and we are left to show those of the $M_i^{(0)}$. Since $\chi\langle\sigma\rangle^{\alpha}$, $\chi_0\langle\sigma_0\rangle^{\alpha}$ (and $J$) are bounded, we need to show that

$$\begin{aligned}
&H\chi(H+i)^{-1}\langle\sigma\rangle^{\alpha}\\
&\qquad = H\chi[\langle\sigma\rangle^{\alpha}(H+i)^{-1} + (H+i)^{-1}[\langle\sigma\rangle^{\alpha}\,,\, H](H+i)^{-1}]
\end{aligned}$$

is, too (and similarly for the '0'-version). Indeed, for $\alpha < 1$, $[\langle\sigma\rangle^{\alpha}\,,\, H](H+i)^{-1}$ is bounded, cf. (3.4), and so is

$$Hf(H+i)^{-1} = H(H+i)^{-1}\big(f + [H\,,\, f](H+i)^{-1}\big)$$

for $f = \chi\langle\sigma\rangle^{\alpha}$ or $f = \chi$.

2. Since $(1-JJ^*)(1-\chi) = 0$ and $\chi\langle\sigma\rangle^{\alpha}$ is bounded, we may show

$$\lim_{t\to\pm\infty}\langle\sigma\rangle^{-\alpha}e^{-iHt}E_{\Delta}(H)\psi = 0.$$

As a function of $t$, this state has bounded derivative and is square integrable in $t$, cf. (3.2). Hence the claim. $\square$

It remains to show that $W_\pm = W_\pm(J)$ is independent of $j$ and $w$ in the construction (1.10) of $J$. We may choose $\tilde{\jmath}$, $\tilde{w}$ still satisfying the requirements (1.11, 1.7) and, moreover,

$$\operatorname{supp}\tilde{\jmath}(u-\tilde{w}(s)) \subset \Omega_-^e \cup \Omega_+^e, \tag{3.12}$$
$$\tilde{\jmath}(u-\tilde{w}(s))j(u-w(s)) = \tilde{\jmath}(u-\tilde{w}(s)) \tag{3.13}$$

for any two given choices $j = j_i$, $w = w_i$, $(i = 1, 2)$. To show $W_\pm(J_1) = W_\pm(J_2)$ it thus suffices to prove $W_\pm(J) = W_\pm(\tilde{J})$ for $J = J_1, J_2$. Since $(s, u)$ are Euclidean coordinates in $\Omega_\pm^e$, eqs. (3.12, 3.13) imply $\tilde{J}\tilde{J}^*J = \tilde{J}$ and therefore

$$\begin{aligned}\underset{t\to\pm\infty}{\text{s-lim}}\,(J-\tilde{J})e^{-iH_0t}E_\Delta(H_0) &= \underset{t\to\pm\infty}{\text{s-lim}}\,(1-\tilde{J}\tilde{J}^*)Je^{-iH_0t}E_\Delta(H_0)\\ &= \underset{t\to\pm\infty}{\text{s-lim}}\,(1-\tilde{J}\tilde{J}^*)e^{-iHt}E_\Delta(H)W_\pm(J) = 0\end{aligned}$$

by (3.9), proving the claim.

# 4 The scattering matrix at large magnetic fields

At large magnetic fields the scattering operator acquires a universal behavior, depending only on the bending angle, but independent of other geometric properties of the domain, as stated in Thm. 2. The estimate (1.21), from which the full statement of the theorem follows by density, will be established through an approximation to the evolution $e^{-iHt}\psi$ which is accurate at all times and not just near $t = \pm\infty$, as was the case in the previous section. To this end we choose an adapted *gauge* and interpret $H$ on $L^2(\Omega)$ as a *perturbation* of $H_0$ on $L^2(\Omega_0)$. This will require an *identification* of the two spaces which is more accurate than (1.10). Since these steps are intended for the limit $\beta \to \infty$, we will assume $\beta \geq 1$ throughout this section.

We begin with the choice of *gauge*, which is a deformation of Landau's.

**Lemma 3.** *There is a smooth vector field on $\Omega$ with $\nabla \wedge A = 1$ and (1.4, 1.8) whose pull-back on $\Omega_0^e$ under the tubular map, $A_0 := (\mathrm{D}\mathcal{T})^t A$, is*

$$A_0(s, u) = -(u - \frac{u^2}{2}\kappa(s), 0). \tag{4.1}$$

In the definition (1.18) of the scattering operator $S$ asymptotic states are represented as states in $\int^\oplus \mathcal{H}_T\,\mathrm{d}k$ by means of $\mathcal{F}_\beta$, see (1.14). It is useful

to make the band structure of $\widehat{H}_0$ explicit there. The range of $E_\Delta(H_0)$ then becomes isomorphic to the direct sum

$$E_\Delta(H_0)\mathcal{H}_0 \cong \bigoplus_{n\in\mathcal{B}} L^2(I_n, \mathrm{d}k),$$

where $I_n := E_n^{-1}[\Delta]$ is bounded and $\mathcal{B} := \{n \in \mathbb{N} \,|\, I_n \neq \emptyset\}$ is finite if $\Delta$ is as in Thm. 2. The isomorphism is established by the unitary

$$\mathcal{U} : \bigoplus_{n\in\mathcal{B}} L^2(I_n, \mathrm{d}k) \;\to E_\Delta(H_0)\mathcal{H}_0, \qquad \mathcal{U} = \bigoplus_{n\in\mathcal{B}} U_n$$

with

$$U_n : \; L^2(I_n, \mathrm{d}k) \;\to E_\Delta(H_0)\mathcal{H}_0, \qquad U_n f := \mathcal{F}_\beta(\psi_n f),$$

i.e.,

$$(U_n f)(s) = \frac{\beta^{1/2}}{(2\pi)^{1/2}} \int\limits_{I_n} e^{i\beta k s} \mathcal{D}_\beta \psi_n(k) f(k) \, \mathrm{d}k. \tag{4.2}$$

The Hamiltonian for the $n$-th band, $U_n^* H_0 U_n =: h_n$, is multiplication by $E_n(k)$. We define *single band wave operators* as

$$\Omega_\pm(n) := \operatorname*{s-lim}_{t\to\pm\infty} e^{iHt} J U_n e^{-ih_n t} = W_\pm U_n, \tag{4.3}$$

and corresponding scattering operators as

$$\sigma_{nm} := \Omega_+^*(n)\Omega_-(m).$$

At this point (1.21) reduces to

$$\left\| \sigma_{nm} - \delta_{nm} e^{i\phi_n(k)} \right\|_{\mathcal{L}(L^2(I_m), L^2(I_n))} \le C_{\Delta,\varepsilon} \beta^{-1+\varepsilon}. \tag{4.4}$$

An improved *identification* operator $\tilde{J} : \; L^2(\Omega_0) \;\to\; L^2(\Omega)$ is

$$(\tilde{J}\psi)(x) = \begin{cases} j(u - w(s)) g(s,u)^{-1/4} \psi(s,u), & \text{if } x = x(s,u) \in \Omega^e, \\ \quad 0, & \text{otherwise.} \end{cases} \tag{4.5}$$

It is obtained as a modification of (1.10), where $g(s,u)^{1/2} = |\det \mathrm{D}\mathcal{T}|$ and $g\mathrm{d}s\mathrm{d}u$ is the Euclidean volume element $\mathrm{d}x_1\mathrm{d}x_2$ in tubular coordinates. We take the parameter $w_0$ in (1.11) so that $3w_0 < \inf_s w(s)$. Then $j(u - w(s)) =$

1 for $u < w_0$ and $\tilde{J}$ acts as an isometry on states supported near $\partial\Omega_0$, which is where we expect edge states to be concentrated at all times.

The *perturbation* induced by the curvature of $\partial\Omega$ on the dynamics will be accounted for by a modification $\tilde{U}_n$ of $U_n$ in (4.2), resp. $\tilde{\mathcal{J}}_n := \tilde{J}\tilde{U}_n$ of $JU_n$ in (4.3):

$$\tilde{U}_n : L^2(I_n, \mathrm{d}k) \to \mathcal{H}_0 \tag{4.6}$$

$$(\tilde{U}_n f)(s) := \frac{\beta^{1/2}}{(2\pi)^{1/2}} \int_{I_n} e^{i(\beta k s + \phi_n(s,k))} \mathcal{D}_\beta \tilde{\psi}_n(s,k) f(k)\,\mathrm{d}k, \tag{4.7}$$

where

$$\phi_n(s,k) = -\frac{E_n^{(1)}(k)}{E_n'(k)} \int_{-\infty}^{s} \kappa(s')\,\mathrm{d}s', \tag{4.8}$$

$$\tilde{\psi}_n(s,k) = \psi_n(k) + \beta^{-1}\kappa(s)\tilde{\psi}_n^{(1)}(k),$$

$$\tilde{\psi}_n^{(1)}(k) = \psi_n^{(1)}(k) - \frac{E_n^{(1)}(k)}{E_n'(k)}\big((\partial_k\psi_n)(k) + \langle(\partial_k\psi_n)(k)\,,\,\psi_n(k)\rangle\psi_n(k)\big),$$

$$\psi_n^{(1)}(k) = -(H_0(k) - E_n(k))^{-1}(1 - P_n(k))H_1(k)\psi_n(k). \tag{4.9}$$

It will be proved later that (4.7) yields a bounded map (4.6). Here we remark that $H_1(k)\psi_n(k)$ is well-defined because $\psi_n(k)$ decays exponentially in $u$ and that $\tilde{\psi}_n(k)$ transforms as (1.17) under a change of phase. A semiclassical interpretation of the above construction is in order. The evolution would adiabatically promote a particle from the asymptotic state $\psi_n(k)$ at $s = -\infty$ to the perturbed eigenstate $\psi_n^{[1]}(s,k) = \psi_n(k) + \beta^{-1}\kappa(s)\psi_n^{(1)}(k)$ of (1.22), if $k$ were an adiabatic invariant. It is only approximately so, since it changes by $\mathrm{d}k/\mathrm{d}t = \{H(s,k), k\} \approx -\beta^{-1}\dot{\kappa}(s)E_n^{(1)}(k)$ per unit time or, cumulatively w.r.t. arc length, by $\delta k(s) = -\beta^{-1}\kappa(s)E_n^{(1)}(k)/E_n'(k)$. Therefore a more accurate state is $e^{i\beta^{-1}\gamma_{\mathrm{B}}(s,k)}\psi_n^{[1]}(s, k + \delta k(s))$, where the phase is determined by parallel transport, see eq. (1.23). For small $\beta^{-1}$ it equals $\tilde{\psi}_n(s,k) + \mathrm{O}(\beta^{-2})$.

The main intermediate result of this section is that $\tilde{J}_n e^{-ih_n t}$ is an accurate approximation of $e^{-iHt}$ at all times in the relevant energy range:

**Proposition 1.** *For all $\varepsilon > 0$ and $\Delta$ as in Thm. 1:*

$$\sup_{t\in\mathbb{R}} \Big\| E_\Delta(H)(e^{-iHt}\tilde{\mathcal{J}}_n - \tilde{\mathcal{J}}_n e^{-ih_n t}) \Big\|_{\mathcal{L}(L^2(I_n), L^2(\Omega))} \le C_{\Delta,\varepsilon}\beta^{-1+\varepsilon}.$$

The implication of this result on the scattering operators $\sigma_{nm}$ can now be phrased conveniently in terms of Isozaki-Kitada wave operators $\tilde{\Omega}_\pm(n)$:

**Proposition 2.** *The limits*

$$\tilde{\Omega}_\pm(n) = \operatorname*{s-lim}_{t\to\pm\infty} e^{iHt}\tilde{\mathcal{J}}_n e^{-ih_n t} \tag{4.10}$$

*exist and equal*

$$\tilde{\Omega}_-(n) = \Omega_-(n), \quad \tilde{\Omega}_+(n) = \Omega_+(n)e^{i\phi_n(k)}. \tag{4.11}$$

*Moreover, for* $\varepsilon > 0$,

$$\left\|\tilde{\Omega}_+^*(n)\tilde{\Omega}_-(m) - \delta_{nm}\right\| \le C\beta^{-1+\varepsilon}. \tag{4.12}$$

Since $\sigma_{nm} = e^{i\phi_n(k)}\tilde{\Omega}_+^*(n)\tilde{\Omega}_-(m)$, the proof of eq. (4.4) and of Thm. 2 is complete, except for the proofs of Lemma 3 and Props. 1, 2 which we will give in the rest of this section.

*Proof.* (Lemma 3) We may first define $A(x)$ for $x \in \Omega_e$ so that (4.1) holds, i.e., in terms of forms $A = (\mathcal{T}^*)^{-1}A_0$, $A_0 = -(u - \frac{u^2}{2}\kappa(s))\mathrm{d}s$. We indeed have $\nabla\wedge A = 1$ there, because

$$\mathrm{d}A_0 = -(1-u\kappa(s))\mathrm{d}u\wedge\mathrm{d}s = g^{1/2}\mathrm{d}s\wedge\mathrm{d}u,$$

and thus $\mathrm{d}A = (\mathcal{T}^*)^{-1}(\mathrm{d}A_0) = \mathrm{d}x_1\wedge\mathrm{d}x_2$, but also $\mathrm{d}A = (\nabla\wedge A)\mathrm{d}x_1\wedge\mathrm{d}x_2$. We also note that (1.4) holds, since $A(\gamma(s))\cdot\dot{\gamma}(s) = A_0(\partial_s)|_{u=0} = 0$. The definition of $A$ can then be extended as follows to all of $\Omega$: Starting from any field $\tilde{A}$ with $\nabla\wedge\tilde{A}\equiv 1$ on $\Omega$ , there is $\chi(x)$ such that $A = \tilde{A} + \nabla\chi$ on $\Omega_e$. Now it suffices to extend the scalar function $\chi$ to $\Omega$. $\square$

Some of the further analysis is conveniently phrased in terms of pseudodifferential calculus, of which we shall need a simple version. We fix a band $n$ with momentum interval $I_n$ and drop the band index $n$ from all quantities throughout the remainder of this section. The symbols are defined on the phase space $\mathbb{R}\times I \ni (s,k)$ of a particle on the boundary $\partial\Omega$ and take values in some Banach space $X$, typically $X\subset\mathcal{H}_T$:

$$\mathcal{A}_2(X) := \{a\,|\,a(s,k)\in X,\ \|a\|^2_{\mathcal{A}_2(X)} := \int\sup_{k\in I}\|a(s,k)\|_X^2\,\mathrm{d}s < \infty\}. \tag{4.13}$$

We abbreviate $\mathcal{A}_2 \equiv \mathcal{A}_2(\mathcal{H}_T)$. If $X = \mathcal{D}(M)$ is the domain of some closed operator $M$ equipped with the graph norm $\|\cdot\|_M = \|\cdot\|_{\mathcal{H}_T} + \|M\cdot\|_{\mathcal{H}_T}$, we just write $\mathcal{A}_2(M) \equiv \mathcal{A}_2(\mathcal{D}(M))$.

For a symbol $a \in \mathcal{A}_2(X)$, we define an operator by *left-quantization*

$$\begin{aligned} \mathrm{Op}(a): \quad & L^2(I) \to L^2(\mathbb{R}, X), \\ (\mathrm{Op}(a)f)(s) &:= \frac{\beta^{1/2}}{\sqrt{2\pi}} \int_I e^{i\beta ks} (\mathcal{D}_\beta a)(s,k) f(k)\, \mathrm{d}k, \end{aligned} \tag{4.14}$$

where $\mathcal{D}_\beta$ is as in (1.15). The integral is a Bochner integral on $\mathcal{H}_T$ [1, Thm. 1.1.4]. It exists pointwise for each $s \in \mathbb{R}$ with $\sup_{k\in I} \|a(s,k)\|_X < \infty$, because $\mathcal{H}_T$ is separable and $\|f\|_1 \le |I|^{1/2}\|f\|_2$. Moreover, (4.14) defines a bounded operator $\mathrm{Op}(a) : L^2(I) \to L^2(\mathbb{R}, X)$, because of

$$\|\mathrm{Op}(a)\| \le \frac{(\beta|I|)^{1/2}}{\sqrt{2\pi}} \|a\|_{\mathcal{A}_2(X)} \|f\|_2.$$

We shall extend in two ways the class of symbols $a$ admissible in (4.14). First, that equation defines a bounded operator $L^2(I) \to L^2(\mathbb{R}, X)$ also if $a(s,k)$ tends to some asymptotes for some $a_\pm(k)$ at large $s$, in the sense that

$$\int_0^{\pm\infty} \sup_{k\in I} \|a(s,k) - a_\pm(k)\|_X^2 \, \mathrm{d}s < \infty, \qquad \sup_{k\in I} \|a_\pm(k)\|_X < \infty.$$

We denote such symbols by $a \in \mathcal{A}(X)$. In fact, the integral is still defined pointwise as before; in the case that $a$ is independent of $s$ the result follows by the unitarity of the Fourier transform, and in general from $a(s,k) - \theta(s)a_+(k) - \theta(-s)a_-(k) \in \mathcal{A}_2(X)$. (Further conditions for $\|\mathrm{Op}(a)\| < \infty$, which we shall not need, are given by the Calderón-Vaillancourt theorem [11].) Second, the notation (4.14) shall be used also when the symbol $a(s,k)$ is actually a polynomial in $\beta^{-1}$, $a(s,k) = \sum_{j=0}^{\deg a} \beta^{-j} a_j(s,k)$, in which case $\|a(s,k)\|_X^2 := \sum_{j=0}^{\deg a} \|a_j(s,k)\|_X^2$. An example for both extensions is $a(s,k) := \tilde\psi(s,k)e^{i\phi(s,k)} \in \mathcal{A}(H_0(k))$, for which $\mathrm{Op}(a) = \tilde U$. In particular (4.7) defines a bounded map, as claimed. Note that $\mathcal{D}(H_0(k))$, see (1.16), is independent of $k$.

The following propagation estimate holds:

**Lemma 4.** *Let $a \in \mathcal{A}_2$. Then*

$$\int_{-\infty}^{\infty} \left\|\mathrm{Op}(a) e^{-iht} f\right\|^2 \mathrm{d}t \le C\beta \|f\|^2, \tag{4.15}$$

*where*

$$C = \int \sup_{k \in I} \frac{\|a(s,k)\|^2}{E'(k)} \, \mathrm{d}s < \infty.$$

*Moreover,*

$$\operatorname*{s-lim}_{t \to \pm\infty} \mathrm{Op}(a) e^{-iht} = 0. \tag{4.16}$$

*Proof.* The integrand of the l.h.s of (4.15) is

$$\left\| \mathrm{Op}(a) e^{-iht} f \right\|^2 = \frac{\beta}{2\pi} \int \mathrm{d}s \int_I \mathrm{d}k_2 \int_I \mathrm{d}k_1 \, e^{i\beta(k_1-k_2)s} e^{-i(E(k_1)-E(k_2))t} \\ \times \langle a(s,k_2)\,,\, a(s,k_1) \rangle \bar{f}(k_2) f(k_1),$$

where we used that $\mathcal{D}_\beta$ is unitary. Formally, we may use

$$\frac{1}{2\pi} \int e^{-i(E(k_1)-E(k_2))t} \, \mathrm{d}t = \delta(E(k_1) - E(k_2)) = E'(k_1)^{-1} \delta(k_1 - k_2),$$

because $k \mapsto E(k)$ is monotonous, so that (4.15) equals

$$\beta \int \mathrm{d}s \int_I \mathrm{d}k \, E'(k)^{-1} \|a(s,k)\|^2 \|f(k)\|^2, \tag{4.17}$$

from which the first claim follows. More carefully, we change variables $k_i \mapsto E(k_i) = e_i$, $\mathrm{d}k_i = E'(k_i)^{-1} \mathrm{d}e_i$ and extend the integrand by zero for $e_i \notin E^{-1}(I)$. Then (4.17) follows by Tonelli's theorem and Parseval's identity.

Eq. (4.16) follows from the fact that $\mathrm{Op}(a) e^{-iht} f$ has bounded derivative in $t$ and is square integrable w.r.t $t$. $\square$

Prop. 1 states that $\tilde{\mathcal{J}} = \tilde{J}\tilde{U}$ approximately intertwines between the dynamics $h$ on $L^2(I, \mathrm{d}k)$ and $H$ on $\mathcal{H}$. Its proof will combine the intertwining properties of $\tilde{J}$ and of $\tilde{U}$, as discussed separately by the following two lemmas.

**Lemma 5.** *Let*

$$H_1 := \beta^{-1} \left( 2(\beta u) D_s \kappa D_s - \frac{1}{2} (\beta u)^2 \{\kappa\,,\, D_s\} \right),$$

*where $D_s = -i\beta^{-1}\partial_s + \beta u$. Then for any $1/2 < \alpha \leq 1$:*

$$(H\tilde{J} - \tilde{J}(H_0 + H_1))\tilde{U} = \langle \sigma \rangle^{-\alpha} R \, \mathrm{Op}(b), \tag{4.18}$$

*where $\|b\|_{\mathcal{A}_2} \leq C$ and $\|R\|_{\mathcal{L}(\mathcal{H}_0, \mathcal{H})} \leq C_\alpha \beta^{-2}$.*

**Lemma 6.** *For any $\alpha > 0$ we have:*

$$(H_0 + H_1)\tilde{U} - \tilde{U}h = \langle\sigma_0\rangle^{-\alpha} R\,\mathrm{Op}(b), \tag{4.19}$$

*where $\|b\|_{\mathcal{A}_2} \le C$, $\|R\|_{\mathcal{L}(\mathcal{H}_0)} \le C_\alpha \beta^{-2}$ and $H_1$ as in Lemma 5.*

The first lemma states that on the image of $\tilde{U}$ the Hamiltonian $H$ is a perturbation of the half-plane Hamiltonian $H_0$. The leading part, $H_1$, of this perturbation is formally of order $\beta^{-1}$, because $\beta u$ and $D_s$ are of $O(1)$ on the image of $\tilde{U}$. Since the tangential velocity $i[H_0\,,\,s] = 2\beta^{-1}D_s$ is of order $\beta^{-1}$, the size of $H_1$ is thus inversely proportional to the time $\sim \beta$ (in units of the inverse cyclotron frequency) required by the particle to traverse the bent, i.e., $\operatorname{supp}\kappa$. The cumulated effect is thus of order 1, like the phase (4.8) which by the second lemma accounts for it to leading order. Subleading contributions occurring in either approximation are formally of order $\beta^{-2}$. They may be integrated in time and controlled by means of the propagation estimates in Lemmas 1, 4.

*Proof.* (Proposition 1) Upon multiplication by $e^{iHt}$ the quantity to be estimated is seen to be

$$E_\Delta(H)(e^{iHt}\tilde{\mathcal{J}}e^{-iht} - \tilde{\mathcal{J}}) = i\int_0^t E_\Delta(H)e^{iH\tau}(H\tilde{\mathcal{J}} - \tilde{\mathcal{J}}h)e^{-ih\tau}\,\mathrm{d}\tau. \tag{4.20}$$

We expand

$$H\tilde{\mathcal{J}} - \tilde{\mathcal{J}}h = (H\tilde{J} - \tilde{J}(H_0 + H_1))\tilde{U} + \tilde{J}((H_0 + H_1)\tilde{U} - \tilde{U}h),$$

and insert the two terms on the r.h.s. into (4.20). We use the general fact that

$$\|T\| = \sup\{|\langle\varphi_2\,,\,T\varphi_1\rangle|\,|\,\varphi_i \in \mathcal{H}_i,\ \|\varphi_i\| = 1,\ (i = 1, 2)\}$$

for operators $T : \mathcal{H}_1 \to \mathcal{H}_2$ between Hilbert spaces, and apply the estimates (4.18, 4.19) on the two contributions respectively. For the second term we also use $\tilde{J}\langle\sigma_0\rangle^{-\alpha} = \langle\sigma\rangle^{-\alpha}\tilde{J}$. Together with (3.2, 4.15), we see that the two contributions are bounded in norm by a constant times $\beta^{-2}\cdot\beta^{(1+\varepsilon)/2}\cdot\beta^{1/2} = \beta^{-1+\varepsilon/2}$. $\square$

The proofs of Lemmas 5, 6 are postponed till after that of Proposition 2.

*Proof.* (Proposition 2) Let $F(s \in A)$ be the characteristic function of the set $A \subset \mathbb{R}$. We claim that for any $a \in \mathbb{R}$

$$\operatorname*{s-lim}_{t\to-\infty} F(s \ge -a)\tilde{U}e^{-iht} = 0, \tag{4.21}$$

and similarly for $U$ instead of $\tilde{U}$, as well as for $F(s \le a)$ and $t \to +\infty$. It will be enough to prove (4.21) when acting on $f \in C_0^\infty(I)$.

We then have

$$(\tilde{U}e^{-iht}f)(s) = \frac{\beta^{1/2}}{(2\pi)^{1/2}} \int_I e^{i(\beta ks - E(k)t + \phi(s,k))} \mathcal{D}_\beta \tilde{\psi}(s,k) f(k) \, \mathrm{d}k$$

with

$$\begin{aligned} \frac{\partial}{\partial k}(\beta ks - E(k)t + \phi(s,k)) &= \beta s - E'(k)t - \frac{\mathrm{d}}{\mathrm{d}k}\left(\frac{E^{(1)}(k)}{E'(k)}\right) \int_{-\infty}^{s} \kappa(s') \, \mathrm{d}s' \\ &\ge 1 + \beta|s+a| + \delta|t| \end{aligned} \tag{4.22}$$

for some $\delta > 0$, all $s \ge -a$ and $-t$ large enough. We may pretend that $\tilde{\psi}(s,k)$ is replaced by $\psi(k)$, as the difference is dealt with by (4.16). Since the latter amplitude is independent of $s$, the usual non-stationary phase method (e.g. [13, Thm. XI.14 and Corollary]) may be applied. We obtain (without keeping track of the dependence of constants on $\beta$)

$$\left\| (\tilde{U}e^{-iht}f)(s) \right\|_{\mathcal{H}_T} \le C_l(1 + |s+a| + |t|)^{-l}, \quad (l \in \mathbb{N}, \, s \ge -a),$$

where we also used that $\psi(k) \in C^\infty(I, \mathcal{H}_T)$. As a result,

$$\left\| F(s \ge -a)\tilde{U}e^{-iht}f \right\|^2 \le C_l'(1 + |t|)^{-2l+1},$$

for $-t$ large enough, proving (4.21). As the estimate (4.22) also holds with $\phi(s,k)$ omitted or replaced by $\phi(k) = \phi(s = \infty, k)$, the result applies to $U$ and $Ue^{i\phi(k)}$ as well.

We maintain that (4.21) implies

$$\operatorname*{s-lim}_{t \to -\infty} (U - \tilde{U})e^{-iht} = 0, \tag{4.23}$$

$$\operatorname*{s-lim}_{t \to +\infty} (Ue^{i\phi(k)} - \tilde{U})e^{-iht} = 0, \tag{4.24}$$

$$\operatorname*{s-lim}_{t \to \pm\infty} (J - \tilde{J})Ue^{-iht} = 0. \tag{4.25}$$

Indeed, if $-a < \operatorname{supp} \kappa$, and hence $e^{i\phi(s,k)} = 1$ as well as $\tilde{\psi}(s,k) = \psi(k)$ for $s < -a$, then

$$U - \tilde{U} = F(s \ge -a)(U - \tilde{U})$$

and (4.23) follows from (4.21). Eq. (4.24) is shown similarly by using $\phi(s,k) = \phi(k)$ for $s > \operatorname{supp}\kappa$. Eq. (4.25) follows from $J - \tilde{J} = (J-\tilde{J})F(|s| \leq a)$, since $g(s,u) = 1$ for $(s,u) \in \Omega_0^e$, $|s| \geq a$. Now (4.10, 4.11) are immediate. They follow from the existence of the wave operators (4.3), i.e., $\Omega_\pm(n) = \text{s-lim}_{t\to\pm\infty} e^{iHt} J U e^{-iht}$, by means of (4.25) and of (4.23), resp. (4.24).

Finally, we prove (4.12). Here it is necessary to introduce the band labels again. By the intertwining property of $\tilde{\Omega}_\pm(n)$ between $H$ and $h_n$ we have

$$
\begin{aligned}
&\langle g\,,\, \tilde{\Omega}_+^*(n)\tilde{\Omega}_-(m) f\rangle = \langle \tilde{\Omega}_+(n) g\,,\, E_\Delta(H)\tilde{\Omega}_-(m) f\rangle \\
&= \lim_{t\to\infty} \langle e^{iHt}\tilde{\mathcal{J}}_n e^{-ih_n t} g\,,\, E_\Delta(H) e^{-iHt}\tilde{\mathcal{J}}_m e^{ih_m t} f\rangle \\
&= \lim_{t\to\infty} \langle \tilde{\mathcal{J}}_n e^{-ih_n t} g\,,\, E_\Delta(H) e^{-2iHt}\tilde{\mathcal{J}}_m e^{ih_m t} f\rangle .
\end{aligned}
$$

By Proposition 1 this inner product equals, up to a function of $t$ bounded by $C\beta^{-1+\varepsilon}\|g\|\|f\|$, the expression

$$
\begin{aligned}
&\langle \tilde{\mathcal{J}}_n e^{-ih_n t} g\,,\, E_\Delta(H)\tilde{\mathcal{J}}_m e^{-ih_m t} f\rangle = \langle e^{iHt}\tilde{\mathcal{J}}_n e^{-ih_n t} g\,,\, E_\Delta(H) e^{iHt}\tilde{\mathcal{J}}_m e^{-ih_m t} f\rangle \\
&\overset{t\to+\infty}{\longrightarrow} \langle \tilde{\Omega}_+(n) g\,,\, \tilde{\Omega}_+(m) f\rangle = \langle e^{i\phi_n(k)} g\,,\, \Omega_+^*(n)\Omega_+(m) e^{i\phi_m(k)} f\rangle = \delta_{nm}\langle g\,,\, f\rangle ,
\end{aligned}
$$

proving (4.12). In the last line we used $\Omega_+^*(n)\Omega_+(m) = \delta_{nm}\mathrm{Id}_{L^2(I_m)}$. This follows from $W_+^* W_+ = \mathrm{Id}_{\mathcal{H}_0}$ and $U_n^* U_m = \delta_{nm}\mathrm{Id}_{L^2(I_m)}$. $\square$

It remains to prove Lemmas 5, 6.

An element of pseudodifferential calculus [11] is the symbolic product. We will need the product of an operator valued symbol $h \in \mathcal{A}(\mathcal{L}(X, \mathcal{H}_T))$ with a vector valued one, $a \in \mathcal{A}(X)$, which in the case that $h(s,k)$ is a polynomial in $k$ is defined as

$$
(h\natural a)(s,k) := \sum_{l=0}^{\deg h} \frac{\beta^{-l}}{i^l l!}(\partial_k^l h)(s,k)\cdot(\partial_s^l a)(s,k),
$$

since the sum is then finite. In applications of this product it is understood that $a \in C_s^{\deg h}(\mathcal{A}(X))$, where $a \in C_s^l(\mathcal{A}(X))$ means $\partial_s^j a \in \mathcal{A}(X)$, $0 \leq j \leq l$.

*Proof.* (Lemma 6) Set $a(s,k) := \tilde{\psi}(s,k)e^{i\phi(s,k)} \in \mathcal{A}(H_0(k))$. Then by Lemma 8,

$$
H_0\tilde{U} = H_0\mathrm{Op}(a) = \mathrm{Op}(H_0(k)\natural a), \tag{4.26}
$$

where $H_0(k)$ is given in (1.16). The operator $H_1$ may be written as

$$H_1 = \beta^{-1}\kappa(s)[2(\beta u)(-i\beta^{-1}\partial_s)^2 + 3(\beta u)^2(-i\beta^{-1}\partial_s) + (\beta u)^3] \\ - i\beta^{-2}\dot{\kappa}(s)[2(\beta u)(-i\beta^{-1}\partial_s) + 3/2(\beta u)^2].$$

According to Lemma 7 we have $k^l \natural a \in \mathcal{A}(e^{\lambda u})$, $(l = 0, 1, 2)$, for some $\lambda > 0$. Therefore, by Lemma 8:

$$H_1\tilde{U} = \beta^{-1}\mathrm{Op}(\kappa(s)(H_1\natural a)) + \beta^{-2}\mathrm{Op}(\dot{\kappa}(s)(H_2\natural a)),$$

where

$$H_1(k) = 2uk^2 + 3u^2k + u^3, \qquad H_2(k) = -i(2uk + (3/2)u^2).$$

By evaluating the expression

$$H_0\natural a = \sum_{l=0}^{2} \frac{\beta^{-l}}{i^l l!}(\partial_k^l H_0)(k) \cdot (\partial_s^l a)(s, k) \tag{4.27}$$

we find:

$$H_0\natural a = \tilde{a}_{00} + \beta^{-1}a_{01} + \beta^{-2}\tilde{a}_{02},$$

where $\tilde{a}_{00} = E(k)\tilde{\psi}(s, k)e^{i\phi(s,k)}$, $a_{01} = -\kappa(s)H_1(k)\psi(k)e^{i\phi(s,k)}$ and $\tilde{a}_{02} \in \mathcal{A}_2$ (coefficients with a tilde may themselves contain higher order terms in $\beta^{-1}$). The derivation is as follows: The r.h.s. of (4.27) equals

$$H_0\natural a = \Big[H_0(k)\tilde{\psi}(s, k) + \beta^{-1}(\partial_s\phi(s, k))H_0'(k)\psi(k)\Big]\, e^{i\phi(s,k)} + O(\beta^{-2}). \tag{4.28}$$

The first contribution equals

$$H_0(k)\tilde{\psi}(s, k) = E(k)\tilde{\psi}(s, k) + \beta^{-1}\kappa(s)\left[\frac{E^{(1)}(k)}{E'(k)}H_0'(k) - H_1(k)\right]\psi(k),$$

which follows because (4.9) provides the eigenvector at first order,

$$\big((H_0(k) - E(k)\big)\big(\psi(k) + \beta^{-1}\kappa(s)\psi^{(1)}(k)\big) = \beta^{-1}\kappa(s)\big(E^{(1)}(k) - H_1(k)\big)\psi(k),$$

and from taking the derivative of $(H_0(k) - E(k))\psi(k) = 0$,

$$\big(H_0(k) - E(k)\big)(\partial_k\psi)(k) = E'(k)\psi(k) - H_0'(k)\psi(k).$$

Since $\partial_s \phi = -(E^{(1)}/E')\kappa$ we see that the second term within the square brackets of (4.28) is canceled inside the first one. Hence

$$H_0 \natural a = \Big[ E(k)\tilde{\psi}(s,k) - \beta^{-1}\kappa(s)H_1(k)\psi(k) \Big] \, e^{i\phi(s,k)} + O(\beta^{-2})$$

accounting for $\tilde{a}_{00}$ and $a_{01}$.

$H_1 \natural a$ and $H_2 \natural a$ are evaluated straightforwardly:

$$\begin{aligned} \beta^{-1}\kappa(s)H_1 \natural a &= \beta^{-1}a_{11} + \beta^{-2}\tilde{a}_{12}, \\ \beta^{-2}\dot{\kappa}(s)H_2 \natural a &= \beta^{-2}\tilde{a}_{22}, \end{aligned}$$

where $a_{11}(s,k) = \kappa(s)H_1(k)\psi(k)e^{i\phi(s,k)}$ and $\tilde{a}_{ij}(s,k) \in \mathcal{A}_2$.

Collecting our expansions we get

$$(H_0 + H_1)\tilde{U} = \mathrm{Op}(\tilde{a}_{00}) + \beta^{-1}\mathrm{Op}(a_{01} + a_{11}) + \beta^{-2}\mathrm{Op}(b),$$

where $b \in \mathcal{A}_2$. Since $\mathrm{Op}(\tilde{a}_{00}) = \tilde{U}h$ and $a_{01} + a_{11} = 0$ we conclude that

$$(H_0 + H_1)\tilde{U} - \tilde{U}h = \beta^{-2}\mathrm{Op}(b).$$

We may extract a smooth characteristic function $\chi$ of $\operatorname{supp}\kappa$ from $\mathrm{Op}(b)$. Then (4.19) follows with $R = \beta^{-2}\langle\sigma_0\rangle^{\alpha}\chi(s)$. □

Inspection of the proof shows that derivatives up to $\ddot{\kappa}(s)$ were assumed bounded. This holds true if $\gamma \in C^4$, as assumed in the Introduction.

*Proof.* (Lemma 5) We begin by factorizing (4.18) as

$$(H\tilde{J} - \tilde{J}(H_0 + H_1))\tilde{U} = \langle\sigma\rangle^{-\alpha} \cdot Q \cdot \langle s\rangle^{-1}R_\lambda\langle s\rangle \cdot \langle s\rangle^{-1}e^{\lambda\beta u}(H_0 + i)\tilde{U},$$

where $\lambda > 0$ is picked small, $R_\lambda = e^{\lambda\beta u}(H_0 + i)^{-1}e^{-\lambda\beta u}$, and

$$Q = \langle\sigma\rangle^{\alpha}\left(H\tilde{J} - \tilde{J}(H_0 + H_1)\right)\langle s\rangle e^{-\lambda\beta u}. \tag{4.29}$$

The claim will be established through

$$\|Q\|_{\mathcal{L}(\mathcal{D}(H_0),\mathcal{H}_0)} \le C\beta^{-2}, \tag{4.30}$$

$$\left\|\langle s\rangle^{-1}R_\lambda\langle s\rangle\right\|_{\mathcal{L}(\mathcal{H}_0,\mathcal{D}(H_0))} \le C, \tag{4.31}$$

$$\langle s\rangle^{-1}e^{\lambda\beta u}(H_0 + i)\tilde{U} = \mathrm{Op}(b), \qquad \|b\|_{\mathcal{A}_2} \le C, \tag{4.32}$$

where $\|\cdot\|_{\mathcal{A}_2}$ is the norm in (4.13).

Indeed, (4.31) follows from

$$\langle s\rangle^{-1} R_\lambda \langle s\rangle = R_\lambda - \langle s\rangle^{-1} R_\lambda [H_0\,,\,\langle s\rangle] R_\lambda$$

and $R_\lambda \in \mathcal{L}(\mathcal{H}_0, \mathcal{D}(H_0))$, $\sup_{\beta\geq 1} \|R_\lambda\|_{\mathcal{L}(\mathcal{H}_0,\mathcal{D}(H_0))} < \infty$.

Turning to (4.32), we recall that by (4.26) $(H_0 + i)\tilde{U} = \mathrm{Op}(a)$ with $a \in \mathcal{A}$ (though $a \notin \mathcal{A}_2$, cf. $\tilde{a}_{00}$). For $\lambda$ small enough we have $a \in \mathcal{A}(e^{\lambda u})$ by Lemma 7. We conclude that $b = \langle s\rangle^{-1} e^{\lambda u} a \in \mathcal{A}_2$.

In order to show (4.30), we have to determine how $H\tilde{J}$ acts. For $\varphi \in C_0^\infty(\bar{\Omega}_0)$, $\varphi|_{\partial\Omega_0} = 0$ a direct computation yields:

$$(H\tilde{J}\varphi)(x) = \begin{cases} (g^{-1/4}\tilde{H} j\varphi)(s,u), & x = x(s,u) \in \Omega^e, \\ 0 & \text{otherwise,} \end{cases}$$

where $j = j(u - w(s))$ and $\tilde{H}$ is the differential operator on $\Omega_0^e$

$$\tilde{H} = g^{1/4}\left(g^{-1/2}\tilde{D}_i g^{1/2} g^{ij}\tilde{D}_j\right) g^{-1/4}, \tag{4.33}$$

$$\tilde{D}_s = -i\beta^{-1}\partial_s + \beta u - \frac{\beta u^2}{2}\kappa(s), \qquad \tilde{D}_u = -i\beta^{-1}\partial_u,$$

$$g(s,u) = (1 - u\kappa(s))^2, \qquad g^{ij} = \begin{pmatrix} g^{-1} & 0 \\ 0 & 1 \end{pmatrix}.$$

In (4.33) summation over $i$, $j = s$, $u$ is understood. The expression inside the brackets is the Laplace-Beltrami operator in tubular coordinates on $\Omega_0^e$ associated to the covariant derivative $-i\beta^{-1}\nabla - \beta A$ on $\Omega^e$. Here we used Lemma 3.

Eq. (4.33) has been rearranged in [6, Thm. 3.1] as $\tilde{H} = T + \beta^{-2}V$ with

$$\begin{aligned} T &= \tilde{D}_s g^{-1}\tilde{D}_s - \beta^{-2}\partial_u^2, \\ V(s,u) &= \frac{1}{2} g^{-3/2}\frac{\partial^2\sqrt{g}}{\partial s^2} - \frac{5}{4} g^{-2}\left(\frac{\partial\sqrt{g}}{\partial s}\right)^2 - \frac{1}{4} g^{-1}\left(\frac{\partial\sqrt{g}}{\partial u}\right)^2. \end{aligned} \tag{4.34}$$

Thus,

$$(H\tilde{J}\varphi)(x) = \begin{cases} (g^{-1/4}(T + \beta^{-2}V) j\varphi)(s,u), & x = x(s,u) \in \Omega^e, \\ 0 & \text{otherwise.} \end{cases} \tag{4.35}$$

States of the form

$$\tilde{\psi}(x) = \begin{cases} g^{-1/4}(s,u)\psi(s,u), & x = x(s,u) \in \Omega^e, \\ 0 & \text{otherwise} \end{cases}$$

have norm $\|\tilde{\psi}\|^2 = \int_{\Omega_0^e} |\psi(s,u)|^2 \, \mathrm{d}s\mathrm{d}u \leq \|\psi\|^2$. Since $V(s,u)$ is bounded on $\Omega_0^e$ and of compact support in $s$, its contribution to $Q$ is seen to satisfy (4.30). As for $T$, we write

$$T = (D_s g^{-1} D_s - \beta^{-2}\partial_u^2) - \frac{\beta^{-1}}{2}\{(\beta u)^2 \kappa g^{-1} \, , \, D_s\} + \frac{\beta^{-2}}{4}(\beta u \kappa)^2 g^{-1}. \quad (4.36)$$

We next Taylor expand $g^{-1}$ to first, resp. zeroth order in $u$ in the first two terms,

$$\begin{aligned} g^{-1} &= 1 + 2u\kappa + g^{-1}(3 - 2u\kappa)(u\kappa)^2, \\ &= 1 + g^{-1}(2 - u\kappa)(u\kappa), \end{aligned}$$

and lump the remainders together with the last term of (4.36). These three remainder contributions to (4.35) have compact support in $s$ and are bounded by $\beta^{-2}$ (in the graph norm of $H_0$) after multiplication by $e^{-\lambda\beta u}$, as in (4.29). They thus comply with (4.30). The expanded terms in (4.36) are

$$D_s^2 - \beta^{-2}\partial_u^2 + \beta^{-1}\big(2(\beta u) D_s \kappa D_s - \frac{1}{2}(\beta u)^2 \{\kappa \, , \, D_s\}\big) = H_0 + H_1.$$

All this means that in proving (4.30) we may now pretend that $H\tilde{J}$ is given by (4.35) with $T + \beta^{-2}V$ replaced by $H_0 + H_1$. This is to be compared with

$$(\tilde{J}(H_0 + H_1)\varphi)(x) = \begin{cases} (g^{-1/4} j (H_0 + H_1)\varphi)(s,u), & x = x(s,u) \in \Omega^e, \\ 0 & \text{otherwise.} \end{cases}$$

The resulting commutator is computed as

$$[H_0 + H_1 \, , \, j] = -i\beta^{-1}\{D_i \, , \, \partial_i j\} - i\beta^{-2}\big(2\beta u\{\kappa\partial_s j \, , \, D_s\} - (\beta u)^2 \kappa \partial_s j\big).$$

Its contribution to (4.29) is estimated by a constant times $e^{-\lambda\beta w_0/4}$ thanks to the choice of $w_0$ made in (4.5). Therefore (4.30) is proved. $\square$

# 5 Higher order approximations: Space Adiabatic Perturbation Theory

In this section we give an outlook on higher order approximations of the scattering operator. The central idea is that our approximation should be viewed as an example of *Space Adiabatic Perturbation Theory* [12].

We ultimately aim at the following generalization of Proposition 2:

**Proposition 3.** *For all $l \geq 1$ there exists an identification $\tilde{\mathcal{J}}_n : L^2(I_n) \to \mathcal{H}$ and a phase function $\phi_n^{(l-1)}(k) = \sum_{j=0}^{l-1} \beta^{-j}\phi_j(k)$ such that the limits*

$$\tilde{\Omega}_\pm(n) = \operatorname*{s-lim}_{t\to\pm\infty} e^{iHt}\tilde{\mathcal{J}}_n e^{-ih_n t}$$

*exist and equal*

$$\tilde{\Omega}_-(n) = \Omega_-(n), \quad \tilde{\Omega}_+(n) = \Omega_+(n)e^{i\phi_n^{(l-1)}(k)}.$$

*Moreover, for $\varepsilon > 0$,*

$$\left\|\tilde{\Omega}_+^*(n)\tilde{\Omega}_-(m) - \delta_{nm}\right\| \leq C\beta^{-l+\varepsilon}.$$

If this proposition holds, we have

$$\left\|(S - S_\phi^{(l)})E_\Delta(\hat{H}_0)\right\| \leq C_{\Delta,\varepsilon}\beta^{-l+\varepsilon},$$

where $S_\phi^{(l)} = \int^\oplus \sum_n e^{i\phi_n^{(l-1)}(k)} P_n(k)\,\mathrm{d}k$.

An immediate consequence is:

**Corollary 1.**

$$\forall n \neq m : \quad \|\sigma_{nm}\|_{\mathcal{L}(L^2(I_m),L^2(I_n))} = O(\beta^{-\infty}).$$

This means that interband scattering is strongly suppressed at large $\beta$.

As before the improved identifications are decomposed as $\tilde{\mathcal{J}}_n = \tilde{J}\tilde{U}_n$. The proof of Proposition 2 carries over to that of Proposition 3 if $\tilde{U}_n$ satisfies the following requirements:

$$\operatorname*{s-lim}_{t\to-\infty}(U_n - \tilde{U}_n)e^{-ih_n t} = 0, \tag{5.1}$$

$$\operatorname*{s-lim}_{t\to+\infty}(U_n e^{i\phi_n^{(l-1)}(k)} - \tilde{U}_n)e^{-ih_n t} = 0. \tag{5.2}$$

$$\sup_{t\in\mathbb{R}}\left\|E_\Delta(H)(e^{-iHt}\tilde{\mathcal{J}}_n - \tilde{\mathcal{J}}_n e^{-ih_n t})\right\|_{\mathcal{L}(L^2(I_n),L^2(\Omega))} \leq C_{\Delta,\varepsilon}\beta^{-l+\varepsilon}. \tag{5.3}$$

Complete proofs of the above statements will be given elsewhere [3]. Here we shall only present a heuristic derivation.

$\tilde{J}$ intertwines more accurately between $H$ and the $l$-th order semiclassical approximation $\hat{H}^{(l)}$ of $T + \beta^{-2}V$, where $T$ and $V$ are as in (4.34):

$$H\tilde{J} - \tilde{J}\hat{H}^{(l)} = O(\beta^{-(l+1)}).$$

Semiclassically means that $\mathcal{D}_\beta^{-1}\hat{H}^{(l)}\mathcal{D}_\beta$ can be written as the Weyl quantization of some symbol $H^{(l)}(s,k) = \sum_{j=0}^{L} \beta^{-j} H_j(s,k)$, (here $L = l+2$), where the symbols $H_j(s,k)$ don't depend on $\beta^{-1}$ anymore. Thus the main task is to find $\tilde{U}_n$ such that

$$\hat{H}^{(l)}\tilde{U}_n - \tilde{U}_n h_n = O(\beta^{-(l+1)}).$$

In the last section we invoked the adiabatic nature of the evolution in order to motivate our construction of the approximate intertwiner $\tilde{U}_n$. This property can be exploited more systematically by means of *Space Adiabatic Perturbation Theory* (SAPT) [12], which allows to construct intertwiners $\tilde{U}_n$ at all orders $l$. Such approximations have to be sufficiently explicit of course in order to be of use.

SAPT applies to mixed quantum systems whose Hamiltonian $\hat{H}$ is the quantization of some operator valued semiclassical symbol $H(z) \asymp \sum_{l=0}^{\infty} \varepsilon^l H_l(z)$ w.r.t. some small parameter $\varepsilon$. $z \in \mathbb{R}^{2d}$ is a phase space variable and the Hilbert space is $L^2(\mathbb{R}^d, \mathcal{H}_f)$, where $\mathcal{H}_f$ is some other separable Hilbert space, called the space of fast degrees of freedom. In our case $d = 1$, $z = (s,k)$, $\varepsilon = \beta^{-1}$ and $\mathcal{H}_f = \mathcal{H}_T$. The role of the Hamiltonian $\hat{H}$ is played by $\mathcal{D}_\beta^{-1}\hat{H}^{(l)}\mathcal{D}_\beta$ here.

SAPT associates to each spectral band $\sigma(z)$ of the principal symbol $H_0(z)$ that is separated by a gap from the rest of the spectrum an *effective Hamiltonian* $\hat{\mathfrak{h}}$ that acts on a fixed Hilbert space $L^2(\mathbb{R}^d, \mathcal{K}_r)$, where $\mathcal{K}_r$ can be any Hilbert space isomorphical to $\pi_0(z)\mathcal{H}_f$ for any $z \in \mathbb{R}^{2d}$. Here $\pi_0(z)$ is the spectral projector of $H_0(z)$ that corresponds to $\sigma(z)$. The effective Hamiltonian is the quantization of a semiclassical symbol $\mathfrak{h} \asymp \sum_{l=0}^{\infty} \varepsilon^l \mathfrak{h}_l$. The symbol can be computed explicitly using a recursive scheme. In our case the spectral band $\sigma(z)$ is identified with one of the deformed Landau levels $E_n(k)$. $\pi_0(s,k) \equiv P_n(k)$ is one dimensional and therefore $\mathcal{K}_r \equiv \mathbb{C}$. $\mathfrak{h}$ is a $\mathbb{C}$-valued symbol.

The main results of SAPT imply the following statement:

*The effective Hamiltonian is approximately intertwined with $\hat{H}$ by an isometry*

$$\mathfrak{J} : L^2(\mathbb{R}^d, \mathcal{K}_r) \to L^2(\mathbb{R}^d, \mathcal{H}_f),$$

*i.e.*

$$\hat{H}\mathfrak{J} - \mathfrak{J}\hat{\mathfrak{h}} = O(\varepsilon^\infty). \tag{5.4}$$

*Approximations to $\mathfrak{J}$ can be computed explicitly in terms of its Weyl-symbol to any finite order in $\varepsilon$.*

In our context the physical meaning of this is that at any order in $\beta^{-1}$ the motion of the particle along the boundary is effectively one-dimensional at large $\beta$. It is described to a very good approximation by an effective Hamiltonian on a space with no transverse degree of freedom. The effective Hamiltonian embodies all effects of the transverse degree of freedom on the longitudinal one.

Eq.(5.4) suggests that we split $\tilde{U}_n$ into

$$\tilde{U}_n = \mathcal{D}_\beta \mathfrak{J}^{(l)} \mathfrak{w}^{(l)},$$

where $\mathfrak{J}^{(l)}$ is an approximation of $\mathfrak{J}$ up to order $O(\beta^{-(l+1)})$ and $\mathfrak{w}^{(l)}$ has to intertwine $\hat{\mathfrak{h}}$ and $h_n$ up to order $O(\beta^{-(l+1)})$. This can be accomplished by standard WKB methods. A formal exact intertwiner $\mathfrak{w}$ between $\hat{\mathfrak{h}}$ and $h_n$ is constructed using generalized eigenfunctions of $\hat{\mathfrak{h}}$:

$$(\mathfrak{w}f)(s) = \frac{\beta^{1/2}}{\sqrt{2\pi}} \int\limits_{I_n} B(s,k) e^{i\beta S(s,k)} \, \mathrm{d}k,$$

where formally

$$\hat{\mathfrak{h}} B(s,k) e^{i\beta S(s,k)} = E_n(k) \cdot B(s,k) e^{i\beta S(s,k)},$$
$$\lim_{s\to-\infty} (B(s,k) e^{i\beta S(s,k)} - e^{i\beta k s}) = 0.$$

WKB approximations to $B(s,k)$ and $S(s,k)$ then yield the approximate intertwiner $\mathfrak{w}^{(l)}$.

From $B(s,k)$ and $S(s,k)$ the phase function of the scattering operator is immediate:

$$\phi^{(l-1)}(k) = \lim_{s\to+\infty} \phi^{(l-1)}(s,k),$$

where

$$\phi^{(l-1)}(s,k) = -i \ln B(s,k) + \beta(S(s,k) - ks). \tag{5.5}$$

The above derivation is rather formal. Neither did we show that (5.1), (5.2) hold nor is it clear from the discussion that the error terms are integrable in time along the evolution which is necessary to prove (5.3). The latter seems plausible, however, because we saw in the last section that the correction to the first order approximation of $\tilde{U}_n$ *is* integrable along the evolution.

In fact a closer look at the technical assumptions made in [12] about the symbol $H_0(z)$ reveals that our symbol $H_0(k)$ fails to comply with some of

them. Apart from taking values in the unbounded operators, which causes minor technical complications, it violates the so called *gap condition*. This is a condition on the growth of the symbol $H_0(k)$ with respect to $k$ relative to the growth of the respective gaps between the deformed Landau levels. The condition is used in the general setting of [12] in order to control the *global* behavior of the various symbols w.r.t. the phase space variable $z$. The formal algebraic relationships between them, which are inherently *local*, are not affected. As is pointed out in [12, Sec. 4.5] this does not mean that SAPT is not applicable. It just means that suitable modifications to the general formalism have to be made in order to cover the special case at hand.

Based on these heuristics, SAPT gives us a recipe for computing the scattering phase up to and including order $O(\beta^{-(l-1)})$:

1. Compute $\hat{H}^{(l)}$,
2. Compute $\mathfrak{h}$, the symbol of the effective Hamiltonian, that corresponds to $\hat{H}^{(l)}$ up to and including order $O(\beta^{-l})$, using the formalism of [12].
3. Compute the scattering phase $\phi_n^{(l-1)}(k)$ from a sufficiently accurate WKB approximation of the generalized eigenfunction of $\hat{\mathfrak{h}}$.

Following these steps we find for $\phi^{(1)}(k) = \phi_0(k) + \beta^{-1}\phi_1(k)$, dropping the band index $n$ again,

$$\mathfrak{h}_0 = E(k), \quad \mathfrak{h}_1(s,k) = \kappa(s)E^{(1)}(k),$$
$$\mathfrak{h}_2(s,k) = E^{(1;2)}(s,k) + \kappa^2(s)E^{(2)}(k) - E'(k)(\partial_s\gamma_{RW})(s,k),$$

$$\phi_0(k) = -\frac{E^{(1)}(k)}{E'(k)} \int_{-\infty}^{\infty} \kappa(s')\,\mathrm{d}s',$$

$$\phi_1(k) = -\frac{1}{E'(k)} \int_{-\infty}^{\infty} \left(E^{(1;2)}(s',k) + \kappa^2(s')E^{(2)}(k)\right)\,\mathrm{d}s'$$
$$+\frac{1}{2}\left(\partial_k\left(\frac{E^{(1)}(k)}{E'(k)}\right)^2 + \left(\frac{E^{(1)}(k)}{E'(k)}\right)^2 \cdot \frac{E''(k)}{E'(k)}\right) \cdot \int_{-\infty}^{\infty} \kappa^2(s')\,\mathrm{d}s',$$

where $E^{(1;2)}(s,k) := \langle \psi(k)\,,\, H_2(s,k)\psi(k)\rangle$, $H_2(s,k)$ is the second order symbol of $\hat{H}^{(2)}$, $\gamma_{RW}$ as in (1.24) and $E^{(2)}(k)$ is the second order correction to the eigenvalue $E(k)$ due to the perturbation $H_1(k)$:

$$E^{(2)}(k) := \sum_{m\neq n} \frac{|\langle \psi_n(k)\,,\, H_1(k)\psi_m(k)\rangle|^2}{E_n - E_m}.$$

The phase was computed from the WKB-ansatz (5.5) where $S(s,k)$ has to satisfy the Hamilton Jacobi equation up to order $\beta^{-3}$,

$$\mathfrak{h}(s, \partial_s S(s,k)) - E(k) = O(\beta^{-3}), \tag{5.6}$$

while $B(s,k)$ has to satisfy the amplitude transport equation [10]

$$\partial_s \left[ B(s,k)^2 \cdot \frac{\partial \mathfrak{h}}{\partial k}(s, \partial_s S) \right] = O(\beta^{-2}). \tag{5.7}$$

It is possible to modify the formalism of SAPT as presented in [12] and tailor it to our needs so that we can express $\tilde{U}_n$ at any order as an operator $\mathrm{Op}(a)$. The symbol $a$ is explicit enough as to enable us to prove (5.1), (5.2), (5.3) rigorously by essentially the same methods as in the last section. Moreover, the same formalism allows for a straightforward recursive computation of the scattering phase without reference to the concept of generalized eigenfunctions and their WKB approximations. A full account of this approach would go beyond the scope of this paper and will be presented in [3].

# 6 Appendix

## 6.1 Exponential decay

**Lemma 7.** *Let $I \subset \mathbb{R}$ be a compact interval. For each $n \in \mathbb{N}$ there exists $C < \infty$ such that for small $\lambda \geq 0$ and all $k \in I$:*

1. $$\left\| e^{\lambda u} \psi_n(k) \right\|_{\mathcal{H}_T} \leq C. \tag{6.1}$$

2. $$\left\| e^{\lambda u} \left( \partial_k \psi_n + \langle \partial_k \psi_n\,,\, \psi_n \rangle \psi_n \right) \right\|_{\mathcal{H}_T} \leq C. \tag{6.2}$$

*3.*

$$\left\| e^{\lambda u} \psi_n^{(1)}(k) \right\|_{\mathcal{H}_T} \leq C,$$

*where* $\psi_n^{(1)}(k) = -(H_0(k) - E(k))^{-1}(1 - P_n(k))H_1(k)\psi_n(k)$ *as in* (4.9).

*Proof.* The following norms refer to $\mathcal{H}_T$ or $\mathcal{L}(\mathcal{H}_T)$, as appropriate. By a covering argument we may assume $I$ to be small as needed.

1. Let $\Gamma \subset \rho(H(k))$, $(k \in I)$, be compact. We have

$$\sup_{z \in \Gamma, k \in I} \left\| e^{\lambda u}(H_0(k) - z)^{-1} e^{-\lambda u} \right\| < \infty \tag{6.3}$$

for small $\lambda$. In fact,

$$e^{\lambda u} H_0(k) e^{-\lambda u} = H_0(k) + 2\lambda \partial_u - \lambda^2$$

differs from $H_0(k)$ by a relatively bounded perturbation, and is thus an analytic family for small $\lambda$. Its resolvent, which appears within norms in (6.3), is therefore bounded. This implies

$$\left\| e^{\lambda u} P_n(k) e^{-\lambda u} \right\| < \infty, \tag{6.4}$$

where $\Gamma$ in

$$P_n(k) = \frac{-1}{2\pi i} \oint_\Gamma (H_0(k) - z)^{-1} \, \mathrm{d}z$$

is a contour encircling $E_n(k)$, $(k \in I)$, counterclockwise. Since (6.4) equals $\left\| e^{\lambda u} \psi_n(k) \right\| \left\| e^{-\lambda u} \psi_n(k) \right\| \geq c \left\| e^{\lambda u} \psi_n(k) \right\|$ with $c > 0$, eq. (6.1) follows.

2. We have $\partial_k P_n(k) = |\partial_k \psi_n\rangle\langle\psi_n| + |\psi_n\rangle\langle\partial_k \psi_n|$, so that (6.2) equals

$$\left\| e^{\lambda u}(\partial_k P_n)\psi_n(k) \right\| \leq \left\| e^{\lambda u}(\partial_k P_n) e^{-\lambda u} \right\| \left\| e^{\lambda u} \psi_n(k) \right\|.$$

The claim then follows from (6.1),

$$\partial_k P_n(k) = \frac{1}{2\pi i} \oint_\Gamma (H_0(k) - z)^{-1} (\partial_k H_0(k)) (H_0(k) - z)^{-1} \, \mathrm{d}z,$$

as well as from (6.3) and $e^{\lambda u}(\partial_k H_0) e^{-\lambda u} = \partial_k H_0$.

3. Finally, the last statement follows similarly from the representation of the reduced resolvent

$$(H_0(k) - E(k))^{-1}(1 - P_n(k)) = \frac{1}{2\pi i} \oint_\Gamma (H_0(k) - z)^{-1} (z - E_n(k))^{-1} \, \mathrm{d}z.$$

□

## 6.2 Left-Quantization

**Lemma 8.** *1. Let $T$ be some closed operator with $\mathcal{D}(T) \subset \mathcal{H}_T$. If $a \in \mathcal{A}(T)$ then $(\mathrm{Op}(a)f)(s) \in \mathcal{D}(T)$ $\forall s \in \mathbb{R}$ and*

$$(1 \otimes T)\,\mathrm{Op}(a)f = \mathrm{Op}(\mathcal{D}_\beta^{-1} T \mathcal{D}_\beta a)f.$$

*2. Let $a \in C_s^l(\mathcal{A}(X))$ for some $l \in \mathbb{N}$, where $X \subset \mathcal{H}_T$ with $\|\cdot\|_{\mathcal{H}_T} \leq C\|\cdot\|_X$. Then $k^l \natural\, a \in \mathcal{A}(X)$ and $(\mathrm{Op}(a)f)(s)$ is $l$-times differentiable in $s$ with*

$$(-i\beta^{-1}\partial_s)^l(\mathrm{Op}(a)f) = \mathrm{Op}(k^l \natural a)f.$$

*3. Let $a \in C_s^2(\mathcal{A}(H_0(k)))$. Then $H_0 \natural a \in \mathcal{A}(\mathcal{H}_T)$, and $\mathrm{Op}(a)f \in \mathcal{D}(H_0)$ with*

$$H_0 \mathrm{Op}(a) = \mathrm{Op}(H_0 \natural a).$$

*Proof.* 1. is an immediate consequence of [1, Proposition 1.1.7].

2. The integrand $e^{i\beta ks}\mathcal{D}_\beta a(s,k)f(k)$ of $(\mathrm{Op}(a)f)(s)$ is $l$-times differentiable in $s$ because $a \in C_s^l(\mathcal{A}(X))$. An application of the Leibniz rule yields

$$(-i\beta^{-1}\partial_s)^l(e^{i\beta ks}\mathcal{D}_\beta a(s,k)f(k)) = \mathcal{D}_\beta\left(\sum_{m=0}^{l} \frac{\beta^{-m}}{i^m m!}(\partial_k^m k^l)(\partial_s^m a)(s,k)\right) \times e^{i\beta ks} f(k) = \mathcal{D}_\beta(k^l \natural a)(s,k)e^{i\beta ks}f(k).$$

Clearly $k^l \natural a \in \mathcal{A}(X)$. In particular $\left\|\partial_s^l(e^{i\beta ks}\mathcal{D}_\beta a(s,k)f(k))\right\|_X \leq C|f(k)| \in L^1(I,X)$. The claim now follows by dominated convergence.

3. We have $\mathcal{D}(H_0(k)) \subset \mathcal{D}(D_u^2) \cap \mathcal{D}(u^2)$. Then, by 1., we have that

$$D_u^2 \mathrm{Op}(a)f = \mathrm{Op}(-\partial_u^2 a)f. \tag{6.5}$$

Moreover 2. implies

$$D_s^2 \mathrm{Op}(a)f = \mathrm{Op}(k^2 \natural a)f + 2(\beta u)\mathrm{Op}(k \natural a)f + (\beta u)^2 \mathrm{Op}(a)f,$$

where $k^l \natural a \in \mathcal{A}(H_0(k))$, $(l = 0,1,2)$. Since $\mathcal{A}(H_0(k)) \subset \mathcal{A}(u^2)$, 1. implies that the r.h.s. of the last equation equals

$$\mathrm{Op}((k^2 \natural a) + 2u(k \natural a) + u^2 a)f = \mathrm{Op}((k+u)^2 \natural a)f. \tag{6.6}$$

Combining (6.5),(6.6) we find

$$H_0 \mathrm{Op}(a)f = (D_u^2 + D_s^2)\mathrm{Op}(a)f = \mathrm{Op}(-\partial_u^2 a + (k+u)^2 \natural a)f = \mathrm{Op}(H_0 \natural a)f.$$

□

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